Indian Statistical Institute
First Semester Examination 2004-2005
B.Math (Hons.) I Year
Analysis I

Time: 3 hrs Date:24-11-04 Maximum Marks: 100

1. Show that following sequences converge as $n \to \infty$, and find their limits.

   (i) $a_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}$ , $n \geq 1$;

   (ii) $b_n = \sqrt{n}(\sqrt{n+1} - \sqrt{n})$ , $n \geq 1$. [10]

2. Show that there is no continuous function $f : I\!R \to I\!R$ such that $f(x) = c$ has exactly two solutions for every $c \in I\!R$. Carefully write down a discontinuous function $f : I\!R \to I\!R$ having this property. [20]

3. Fix $-\infty < a < b < \infty$. Let $g : [a,\ b] \to I\!R$ be an increasing function. Suppose $g$ maps $[a,\ b]$ onto $[g(a),\ g(b)]$. Show that $g$ is continuous on $[a,\ b]$. Give an example to show that this need not hold if $g$ is not increasing. [20]

4. Let $f : [a, b] \to I\!R$ be monotonic, and let $\alpha : [a, b] \to I\!R$ be increasing and continuous. Then show that $f$ is Riemann-Stierltjes integrable with respect to $\alpha$. [20]

5. Let $h : [0, 1] \to I\!R$ be a continuous function. Suppose $h(x) \geq 0$ for all $x \in [0, 1]$ and $\int_0^1 h(x)dx = 0$. Then show that $h(x) = 0$ for all $x \in [0, 1]$. [10]

6. State Tayler's theorem. Use it to prove the binomial theorem in the form:
$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots + x^n,$$
   for $n \geq 1$ and $x \in I\!R$. [20]

7. Show that $g : I\!R \to I\!R$ defined by
$$g(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$
   [10]

   is differentiable everywhere, but the derivative is not continuous at 0.